Heat Conduction from Donuts

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Abstract—In this paper, an exact solution of steady heat conduction from a hot donut (a torus) placed in an infinite medium of constant temperature is obtained. The governing energy equation is recast in a naturally-fit coordinates system and then solved using toroidal basis functions.

Index Terms—Donuts, heat conduction, toroidal coordinates.

I. INTRODUCTION

Several industries involve cooling of ring-like blanks (donuts). In such industries, which are not limited to food processing, it is very important to estimate, for example, the cooling time of such products before packaging.

The problem considered here is that of a torus (donut) with inner radius r and outer radius R , heated to a uniform temperature T_s and then placed in a medium of constant temperature T_f , Fig. 1. Our interest is to find the temperature distribution around the donut.

II. METHOD OF SOLUTION

A torus is generated by revolving a circle in three dimensional space about a coplanar axis which does not necessarily touch the circle. If the axis touches the circle, the resulting surface is a horn torus. Furthermore, if the axis is a chord of the circle, the resulting surface is a spindle torus. A sphere is the degenerate case when the axis is a diameter of the circle [1]– [3].

To suit the geometry of the problem, we use the Toroidal Coordinate System. The toroidal coordinate system (η, ξ, ϕ) , is related to the cartesian coordinate system by the relations

$$\begin{aligned} x &= \frac{c \sinh \xi \cos \phi}{\cosh \xi - \cos \eta}, & y &= \frac{c \sinh \xi \sin \phi}{\cosh \xi - \cos \eta}, \\ z &= \frac{c \sin \eta}{\cosh \xi - \cos \eta} \end{aligned} \quad (1)$$

with the corresponding scale factors given by:

$$h_\xi = h_\eta = \frac{c}{\cosh \xi - \cos \eta}, \quad h_\phi = \frac{c \sinh \xi}{\cosh \xi - \cos \eta} \quad (2)$$

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Fig. 1. Problem configuration.

Fig. 2. Toroidal coordinates.

The coordinates satisfy $\xi \in [0, \infty)$, $\eta \in [0, 2\pi)$, and $\phi \in [0, 2\pi)$ with c ($c = \sqrt{R^2 - r^2}$) being the focal distance. The toroidal coordinate system is composed of surfaces of constant ξ which are given by the toroids $x^2 + y^2 + z^2 + c^2 = 2c\sqrt{x^2 + y^2} \coth \xi$, surfaces of constant η given by the spherical bowls $x^2 + y^2 + (z - c \cot \eta)^2 = c^2 / \sin^2 \eta$, and surfaces of constant ϕ given by the half planes $\tan \phi = y/x$, Fig. 2.

The steady version of the differential equation of heat conduction for a homogeneous isotropic solid with no heat

generation is given by

$$\nabla^2 T = 0 \quad (3)$$

where T is the temperature, and α is the thermal diffusivity. In a general orthogonal curvilinear coordinates system (u_1, u_2, u_3) , equation (3) with scale factors h_1, h_2 and h_3 takes the form

$$\frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial T}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial T}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial T}{\partial u_3} \right) \right] = 0 \quad (4)$$

Equation (4) may be specialized for toroidal coordinates as

$$\frac{\partial}{\partial \xi} \left(\frac{\sinh \xi}{\cosh \xi - \cos n} \frac{\partial T}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{\sinh \xi}{\cosh \xi - \cos n} \frac{\partial T}{\partial \eta} \right) = 0 \quad (5)$$

The boundary conditions to be satisfied are:

$$T(\eta, \xi_0) = T_s, \quad T(0, 0) = T_f \quad (6)$$

where ξ_0 ($\xi_0 = \cosh^{-1} \left(\frac{R}{r} \right)$) defines the surface of the torus.

The point $(\eta, \xi) \rightarrow (0, 0)$ defines the far field away from the torus whereas the first condition in (6) indicates that the surface of the torus is maintained at constant temperature.

We define the dimensionless temperature as

$$u(\eta, \xi) = \frac{T - T_f}{T_s - T_f} \quad (7)$$

Equation (5) and the boundary conditions (6) can now be rewritten in terms of the dimensionless temperature as

$$\frac{\partial}{\partial \xi} \left(\frac{\sinh \xi}{\cosh \xi - \cos n} \frac{\partial u}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{\sinh \xi}{\cosh \xi - \cos n} \frac{\partial u}{\partial \eta} \right) = 0 \quad (8)$$

$$u(\eta, \xi_0) = 1, \quad u(0, 0) = 0 \quad (9)$$

Laplace equation (the energy equation (8)) is R-separable in toroidal coordinates, [4-6]. We, therefore, assume a solution of the form

$$u(\eta, \xi) \equiv \sqrt{\cosh \xi - \cos \eta} X(\xi) Y(\eta) \quad (10)$$

Upon substituting (10) in (8), and performing the necessary rituals, one finds the basis functions are $P_{n-\frac{1}{2}}(\cosh \xi), Q_{n-\frac{1}{2}}(\cosh \xi), \cos n\eta, \sin n\eta$, where $P_{n-\frac{1}{2}}$ and $Q_{n-\frac{1}{2}}$ are the toroidal functions of the first and second kinds respectively. Due to the boundary conditions in (9) and the boundedness requirements, only $P_{n-\frac{1}{2}}$ and $\cos n\eta$ survive.

The solution can then be written as

$$u(\eta, \xi) \equiv \sqrt{\cosh \xi - \cos \eta} \sum_{n=0}^{\infty} A_n P_{n-\frac{1}{2}}(\cosh \xi) \cos(n\eta) \quad (11)$$

We apply the first condition in (9) to get A_n . One may make use of the integral

$$\int_0^{2\pi} \frac{\cos(n\eta)}{\sqrt{\cosh \xi - \cos \eta}} d\eta = 2\sqrt{2} Q_{n-\frac{1}{2}}(\cosh \xi) \quad (12)$$

to show that, [7]

$$A_n = \frac{2\sqrt{2}}{\pi(1+\delta_{n0})} \frac{Q_{n-\frac{1}{2}}(\cosh \xi_0)}{P_{n-\frac{1}{2}}(\cosh \xi_0)} \quad (13)$$

The solution of (11), thus, is

$$u(\eta, \xi) \equiv \sqrt{\cosh \xi - \cos \eta} \times \sum_{n=0}^{\infty} \frac{2\sqrt{2}}{\pi(1+\delta_{n0})} \frac{Q_{n-\frac{1}{2}}(\cosh \xi_0)}{P_{n-\frac{1}{2}}(\cosh \xi_0)} P_{n-\frac{1}{2}}(\cosh \xi) \cos(n\eta) \quad (14)$$

where δ_{ij} is Kronecker delta.

Contour plots of the solution (14) for $r=1$ and $R=3/2, 5/2$, and $7/2$ are shown in Figure 3.

Fig. 3. Dimensionless temperature distribution for the cases $r=1$ and $R=3/2, 5/2$, and $7/2$.

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APPENDIX ORTHOGONAL CURVILINEAR COORDINATES

A Cartesian coordinate system offers the unique advantage that all three unit vectors, $\hat{i}$, $\hat{j}$, and $\hat{k}$, are constant. Unfortunately, not all physical problems are well adapted to solution in Cartesian coordinates. In Cartesian coordinates we deal with three mutually perpendicular families of planes: $x = \text{constant}$, $y = \text{constant}$, and $z = \text{constant}$. We superimpose on this system three other families of surfaces. The surfaces of any one family need not be parallel to each other and they need not be planes. Any point is described as the intersection of three planes in Cartesian coordinates or as the intersection of the three surfaces which form curvilinear coordinates. Describing curvilinear coordinates surfaces by $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ where c_1 , c_2 , and c_3 are constant, we identify a point by three numbers (u_1, u_2, u_3) , called the curvilinear coordinates of the point.

Let the functional relationship between curvilinear

coordinates (u_1, u_2, u_3) and the Cartesian coordinates (x, y, z) be given as:

$$\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned} \quad (A1)$$

which can be inverted as

$$\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned} \quad (A2)$$

Any point P in the domain, having curvilinear coordinates (c_1, c_2, c_3) , there will pass three isotimic surfaces (on which the value of a given quantity is everywhere equal) $u_1(x, y, z) = c_1$, $u_2(x, y, z) = c_2$, and $u_3(x, y, z) = c_3$. As illustrated in Figure A.1, these surfaces intersect in pairs to give three curves passing through P , along each of which only one coordinate varies; these are the coordinates curves.

Fig. A.1. A curvilinear Coordinate System (u_1, u_2, u_3)

The normal to the surface $u_i = c_i$ is the gradient:

$$\vec{\nabla} u_i = \frac{\partial u_i}{\partial x} \hat{i} + \frac{\partial u_i}{\partial y} \hat{j} + \frac{\partial u_i}{\partial z} \hat{k} \quad (A3)$$

The tangent to the coordinate curve for u_i is the vector

$$\frac{\partial \vec{r}}{\partial u_i} = \frac{\partial x}{\partial u_i} \hat{i} + \frac{\partial y}{\partial u_i} \hat{j} + \frac{\partial z}{\partial u_i} \hat{k} \quad (A4)$$

We say that u_1, u_2 , and u_3 are orthogonal curvilinear coordinates whenever the vectors $\vec{\nabla} u_1, \vec{\nabla} u_2$ and $\vec{\nabla} u_3$ are mutually orthogonal at every point.

Each gradient vector $\vec{\nabla} u_i$ is parallel to the tangent vector $\frac{\partial \vec{r}}{\partial u_i}$ for the corresponding coordinate curve. And any coordinate curve for u_i intersects the isotimic surface $u_i = c_i$ at right angles when (u_1, u_2, u_3) form orthogonal curvilinear coordinate. To see this, consider a coordinate curve for u_1 .

1) This curve is the intersection of two surfaces $u_2 = c_2$ and

$u_3 = c_3$. Hence, its tangent $\frac{\partial \vec{r}}{\partial u_1}$ is perpendicular to both

surface normals $\vec{\nabla} u_2$ and $\vec{\nabla} u_3$.

2) The vector $\vec{\nabla} u_1$ is also perpendicular to $\vec{\nabla} u_2$ and $\vec{\nabla} u_3$.

3) This implies that $\frac{\partial \vec{r}}{\partial u_1}$ is parallel to $\vec{\nabla} u_1$.

Since both point in the direction of increasing u_1 , they are parallel. It follows, that the vectors $\frac{\partial \vec{r}}{\partial u_1}$, $\frac{\partial \vec{r}}{\partial u_2}$, and $\frac{\partial \vec{r}}{\partial u_3}$ also form a right-handed system of mutually orthogonal vectors. Define the right-handed system of mutually orthogonal unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by

$$\mathbf{e}_i = \frac{\vec{\nabla} u_i}{|\vec{\nabla} u_i|} = \frac{\frac{\partial \vec{r}}{\partial u_i}}{\left| \frac{\partial \vec{r}}{\partial u_i} \right|} \quad i = 1, 2, 3 \quad (A5)$$

We need three functions h_i known as the scale factors to express the vector operations in orthogonal curvilinear coordinates. The scale factor h_i is defined to be the rate at which the arc length increases on the i^{th} coordinate curve with respect to u_i . In other words, if s_i denotes arc length on the i^{th} coordinate curve measured in the direction of increasing u_i , then

$$h_i = \frac{ds_i}{du_i} \quad i = 1, 2, 3 \quad (A6)$$

Since the arc length can be expressed as

$$ds = |d\vec{r}| = \left| \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 \right| \quad (A7)$$

We see that

$$h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right| \quad i = 1, 2, 3 \quad (A8)$$

Hence

$$h_i^2 = \left(\frac{\partial x}{\partial u_i} \right)^2 + \left(\frac{\partial y}{\partial u_i} \right)^2 + \left(\frac{\partial z}{\partial u_i} \right)^2 \quad i = 1, 2, 3 \quad (A9)$$

Combine equations (A7) and (A8) shows that the displacement vector can be expressed in terms of the scale factors by

$$d\vec{r} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3 \quad (A10)$$

A differential length ds in the rectangular coordinate system (x, y, z) is given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \quad (A11)$$

The differential lengths dx , dy and dz are obtained from

equation (A1) by differentiation

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 + \frac{\partial x}{\partial u_3} du_3 \quad (A12a)$$

$$dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2 + \frac{\partial y}{\partial u_3} du_3 \quad (A12b)$$

$$dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2 + \frac{\partial z}{\partial u_3} du_3 \quad (A12c)$$

Substitute equations (A12) into equation (A11), the differential length ds in the orthogonal curvilinear coordinate system (u_1, u_2, u_3) can be written as

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \quad (A13)$$

In rectangular coordinates system a differential volume element dV is given by

$$dV = dx dy dz \quad (A14)$$

and the differential areas dA_x , dA_y , and dA_z cut from the planes $x = \text{constant}$, $y = \text{constant}$, and $z = \text{constant}$ are given, respectively, by

$$dA_x = dy dz, \quad dA_y = dx dz, \quad dA_z = dx dy \quad (A15)$$

In orthogonal curvilinear coordinates system, the elementary lengths from equation (A6) are given, by

$$ds_i = h_i du_i, \quad i = 1, 2, 3 \quad (A16)$$

Then, an elementary volume element dV in orthogonal curvilinear coordinates system given, by

$$dV = h_1 h_2 h_3 du_1 du_2 du_3 \quad (A17)$$

The differential areas dA_1 , dA_2 and dA_3 cut from planes $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ are given, respectively, by

$$\begin{aligned} dA_1 &= ds_2 ds_3 = h_2 h_3 du_2 du_3 \\ dA_2 &= ds_1 ds_3 = h_1 h_3 du_1 du_3 \\ dA_3 &= ds_1 ds_2 = h_1 h_2 du_1 du_2 \end{aligned} \quad (A18)$$

The gradient is a vector having the magnitude and direction of the maximum space rate of change. Then, for any function $\psi(u_1, u_2, u_3)$ the component of $\vec{\nabla}\psi(u_1, u_2, u_3)$ in the $\mathbf{e}_1$ direction is given, by

$$\vec{\nabla}\psi|_1 = \frac{d\psi}{ds_1} = \frac{\partial\psi}{h_1 \partial u_1} \quad (A19)$$

Since this is the rate of change of ψ with respect to distance in the $\mathbf{e}_1$ direction. Since $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are mutually orthogonal unit vector, the gradient becomes

$$\begin{aligned} \text{grad}\psi &= \vec{\nabla}\psi(u_1, u_2, u_3) = \frac{d\psi}{ds_1} \mathbf{e}_1 + \frac{d\psi}{ds_2} \mathbf{e}_2 + \frac{d\psi}{ds_3} \mathbf{e}_3 \\ &= \frac{\partial\psi}{h_1 \partial u_1} \mathbf{e}_1 + \frac{\partial\psi}{h_2 \partial u_2} \mathbf{e}_2 + \frac{\partial\psi}{h_3 \partial u_3} \mathbf{e}_3 \end{aligned} \quad (A20)$$

Let $\vec{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ be a vector field, given in terms of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Then the divergence of $\vec{F}$, denoted $\text{div } \vec{F}$, or $\vec{\nabla} \cdot \vec{F}$ is given, by [23]

$$\vec{\nabla} \cdot \vec{F}(u_1, u_2, u_3) = \lim_{dV \rightarrow 0} \frac{\int \vec{F} \cdot d\mathbf{A}}{\int dV} \quad (A21)$$

where $\int dV$ is the volume of a small region of space and $d\mathbf{A}$ is the vector area element of this volume.

Fig. A2. Differential Rectangular Parallelepiped

We shall compute the total flux of the field $\vec{F}$ outwards of the small rectangular parallelepiped Figure A2. We, then, divide this flux by the volume of the box and take the limit as the dimensions of the box go to zero. This limit is the $\vec{\nabla} \cdot \vec{F}$. Note that the positive direction has been chosen so that (u_1, u_2, u_3) or $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ from a right-hand system.

The area of the face $abcd$ is $h_2 h_3 du_2 du_3$ and the flux normal is $\vec{F} \cdot \mathbf{e}_1 = F_1$. Then the flux outward from the face $abcd$ is $F_1 h_2 h_3 du_2 du_3$. The outward unit normal to the face $efgh$ is $-\mathbf{e}_1$, so its outward flux is $-F_1 h_2 h_3 du_2 du_3$. Since F_1 , h_2 , and h_3 are functions of u_1 as we move along the u_1 -coordinate curve, the sum of these two is approximately

$$\left[\frac{\partial}{\partial u_1} (F_1 h_2 h_3) du_1 \right] du_2 du_3 \quad (A22)$$

Adding in the similar results for the other two pairs of faces, we obtain the net flux outward from the parallelepiped is

$$\left[\frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (F_2 h_1 h_3) + \frac{\partial}{\partial u_3} (F_3 h_1 h_2) \right] du_1 du_2 du_3 \quad (A23)$$

Divide equation (A23) by differential volume equation (A17). Hence the flux output per unit volume is given by

$$\begin{aligned} \operatorname{div} \vec{F} &= \vec{\nabla} \bullet \vec{F} \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (F_2 h_1 h_3) + \frac{\partial}{\partial u_3} (F_3 h_1 h_2) \right] \quad (\text{A24}) \end{aligned}$$

By using equations (A20) and (A24), the Laplacian is given as

$$\begin{aligned} \nabla^2 \psi &= \operatorname{div} \operatorname{grad} \psi \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right] \quad (\text{A25}) \end{aligned}$$

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