Damped Oscillator under Stokesian Realm and Added-Mass Effects

Jiradech Kongthon

Abstract—This article presents the modeling and simulations of a sphere that oscillates vertically in a high-viscosity liquid. The sphere is connected to a linear spring and given an initial displacement from the equilibrium position to allow free vibration and the sphere undergoes the inertia force, the spring force, the drag force, the buoyancy force, the gravity force, and the added-mass force. In general, the added-mass force is not considered in modeling an oscillator. In this article, the added-mass force is included in the modeling to reflect the reality and the effect of the added-mass force is investigated and discussed. The main contribution of this article is to model and simulate the system and to show that i) the natural frequency of oscillation is reduced by the added mass; ii) the damping ratio is also reduced by the added mass; iii) the return time to the original equilibrium is increased by the added mass; iv) furthermore, the difference on the densities between the liquid's density and the sphere's density dictates the degree of the added-mass effect; i.e., the effect due to the added mass is small if the difference on the densities is large.

Index Terms—Added mass, damped oscillator, sphere, stokes' drag, Stokesian regime.

I. INTRODUCTION

In general, the added mass is not considered in modeling an oscillator since the benchmark problem in the study of vibration considers the motion of the object in the air medium, which serves a small amount of the added mass to the body, and the added mass is neglected. In this article, a sphere is set into oscillation in a liquid where the added mass due to the surrounding liquid is significant and needs to be taken into account in the modeling. The previous work in [1] has shown that the added mass due to the surrounding fluid can affect the natural frequency of the membranes.

The main contribution of this article is to model and simulate the system and to show that i) the natural frequency of oscillation is reduced by the added mass; ii) the damping ratio is also reduced by the added mass; iii) the return time to the original equilibrium is increased by the added mass; iv) furthermore, the difference on the densities between the liquid's density and the sphere's density dictates the degree of the added-mass effect; i.e., the effect due to the added mass is small if the difference on the densities is large.

The article provides a brief discussion on necessary backgrounds for modeling in Section II. Section III explains

the modeling using Newtonian mechanics. The simulation using Matlab computing software is in Section IV. The discussion is provided in Section V. Section VI concludes the article.

II. MODELING BACKGROUND

A. Stokesian Dynamics

When a small body moves in a high-viscosity fluid, the motion is assumed to take place in a special dynamical regime which is referred to as the Stokesian realm [2]. If the motion of a small sphere is in this regime, the frictional force that acts on the sphere and opposes the motion of the sphere is linearly proportional to the sphere's velocity and it is known as Stokes' drag. The drag force F_d , is given by Stokes' law, as in [3]; i.e.,

$$F_d = c\dot{x} = 6\pi\mu R\dot{x} \quad (1)$$

where F_d is the drag force (in N), which is known as Stokes' drag, μ is the dynamic viscosity (in Pas), R is the radius of the spherical object (in m), and $\dot{x}$ is the sphere's velocity (in m/s).

The constant c is called the damping constant as the drag force damps out the energy of the moving sphere. The damping constant is given as in (1); i.e.,

$$c = 6\pi\mu R \quad (2)$$

B. Added Mass

The added mass arises in a dynamic system as an object needs to accelerate or decelerate the fluid around it and the mass of the fluid that the object needs to move is considered to be an additional inertia to the object.

The fascinating history, written in [4], of the work on the added mass can be traced back to the time in 1776. Dubuat carried out the experiments on the pendulum oscillating in liquids and he came up with the notion of added mass.

Note that the added mass substantially provides extra inertia to aquatic animals and submarine vehicles when they move. Aquatic animals and underwater vehicles have to generate additional power to account for the mass added from the surrounding water. Some types of fish need to increase the beat frequency of their fins to get extra power to compensate for the added mass. In the design of submarine vehicles, it is very important to determine the submarine added mass coefficients to include in the equations of motion [5], [6].

For a sphere of radius r moving in a liquid of density ρ_l , the added mass m_a is quantified in [7] and it is equal to one half the mass of fluid displaced, or

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$$m_a = \frac{2}{3} \pi r^3 \rho_l \quad (3)$$

III. SYSTEM MODELING

A single-degree-of-freedom system is considered and shown in Fig. 1(a). A sphere of mass m submerged in a high-viscosity liquid of dynamic viscosity μ is suspended vertically by a spring of a spring constant k and the sphere is located in a position far away from the liquid surface and the container surfaces.

In Fig. 1(a), the sphere is in the equilibrium position with the static deformation of the spring of δ . The free-body diagram that shows all of the forces acting on the sphere is shown in Fig. 2(a).

Fig. 1. (a) Sphere at equilibrium, $x=0$ and (b) Sphere moved for a distance of x away from equilibrium.

Fig. 2. (a) Free-body diagram of the sphere at equilibrium, $x=0$ and (b) Free-body diagram of the sphere moved for a distance of x away from equilibrium.

At the equilibrium state, the summation of all the forces in the vertical direction acting on the sphere must be equal to zero, or the sum of the spring force F_s and buoyancy force F_b must equal the weight of the sphere mg acting downward according to the law of gravity.

It is assumed that the spring is linear. For a linear spring, the spring force F_s is directly proportional to the spring deformation δ at the equilibrium position and it is governed by Hooke's law; i.e.,

$$F_s = k\delta$$

where k is a spring constant representing, or quantifying the stiffness of the spring.

According to Archimedes' principle, the buoyancy force F_b always acting upward is given by

$$F_b = \rho_l Vg$$

where ρ_l is the density of the liquid, V is the volume of the displaced fluid, or the volume of the sphere, and g is the gravitational constant.

At the equilibrium position by Newton's first law, it follows that in the vertical direction,

$$\begin{aligned} \Sigma F &= 0, \text{ i.e.,} \\ -k\delta - \rho_l Vg + mg &= 0 \end{aligned}$$

or,

$$mg = k\delta + \rho_l Vg \quad (4)$$

When the sphere is displaced vertically and released, it oscillates with the displacement of x , as in Fig.1(b), from the equilibrium position due to the restoring force from the spring.

Let x or $x(t)$ be the displacement of the sphere from the equilibrium position at time t . It follows that the first time derivative of x or $\dot{x}$ or $\dot{x}(t)$ and the second time derivative of x or $\ddot{x}$ or $\ddot{x}(t)$ represent the velocity and the acceleration of the sphere respectively.

For the sake of convenience, the zero displacement $x = 0$ is set to be the equilibrium position, and the downward direction is chosen as the positive direction. However, both choices are arbitrary.

When the sphere is displaced vertically for an initial displacement x_0 and possibly with an initial velocity $\dot{x}_0$ and released, the sphere is set to allow the oscillation around the equilibrium position ($x = 0$) and the sphere of mass m (with inertia force of $m\ddot{x}$) also needs to accelerate the surrounding liquid of the added mass m_a in addition to its own mass m . There is hence an increase in the inertia force of $m_a\ddot{x}$.

Therefore, the actual inertia force is $(m + m_a)\ddot{x}$.

The spring force F_s at the sphere's position x from the equilibrium position equals

$$F_s = k(\delta + x)$$

In the oscillation mode, the motion is governed by Newton's second law; i.e.,

$$\Sigma F = (m + m_a)\ddot{x} \quad (5)$$

From the free-body diagram shown in Fig. 2(b), (5) becomes

$$mg - k(\delta + x) - \rho_l Vg - c\dot{x} = (m + m_a)\ddot{x} \quad (6)$$

By substituting (4) into (6) and rearranging, (6) becomes

$$(m + m_a)\ddot{x} + c\dot{x} + kx = 0 \quad (7)$$

The damping constant c is determined by (2) assuming the motion occurs in the Stokesian realm.

Rewriting (7) in the standard form leads to

$$\ddot{x} + \frac{c}{m + m_a}\dot{x} + \frac{k}{m + m_a}x = 0 \quad (8)$$

It is known that (8) is, by definition, equivalent to

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0 \quad (9)$$

where ζ denotes the damping ratio and ω_n denotes the natural frequency of oscillation.

The linear ordinary differential equation obtained in (9) indicates that under the Stokesian realm, the system behaves as a linearly damped oscillator.

The natural frequency is determined by matching the coefficients of the variables in (8) and (9) in the third term and as a result, it is given by

$$\omega_n = \sqrt{\frac{k}{m + m_a}} \quad (10)$$

The damping ratio is also obtained by matching the coefficients of the variables in (8) and (9) in the second term and using the result in (10) and it is given by

$$\zeta = \frac{c}{2\sqrt{(m + m_a)k}} \quad (11)$$

Equation (10) shows that with the added mass included in the model to reflect the reality, the natural frequency is decreased.

Equation (11) shows that with the added mass included in the model to reflect the reality, the damping ratio is reduced. This results in the increase in the amplitude of oscillation.

The solution to (9) is known to be, as in [8]

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad (12)$$

where the amplitude A and the phase ϕ are determined from the initial conditions,

$$A = \sqrt{\frac{(\dot{x}_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

$$\phi = \tan^{-1} \frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}$$

The damped natural frequency ω_d is defined by

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

Differentiating the displacement expressed in (12) with

respect to time yields the velocity of the sphere

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + \omega_d Ae^{-\zeta\omega_n t} \cos(\omega_d t + \phi) \quad (13)$$

Differentiating the velocity expressed in (13) with respect to time yields the acceleration of the sphere

$$\ddot{x}(t) = (\zeta^2\omega_n^2 - \omega_d^2)Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) - 2\zeta\omega_n\omega_d Ae^{-\zeta\omega_n t} \cos(\omega_d t + \phi) \quad (14)$$

From (12), the spring force can be found to be

$$f_s = kx \quad (15)$$

From (13), the damping force can be computed

$$f_d = c\dot{x} \quad (16)$$

and from (14), the inertia force can be determined

$$f_m = (m + m_a)\ddot{x} \quad (17)$$

In the modeling without considering the added mass, the quantity m_a in (5), (10), (11), and (17) is equal to zero.

IV. SIMULATION

The dynamic model obtained in Section III is simulated by using Matlab computing software to investigate the effects of the added mass on the dynamic responses such as the natural frequency, the damping ratio, and the time required for the sphere to return to the original equilibrium position to settle down. In addition, the difference on the densities between the liquid's density and the sphere's density is also studied in the simulation.

In the simulations, it is assumed that the sphere has a radius of 0.0025 m; the spring constant is chosen to be 100 N/m; the liquid has the dynamic viscosity of 1 Pas, and the sphere is given an initial displacement of 0.1 m.

The variations in the difference between the sphere's density, ρ_s and the liquid's density, ρ_l are made.

The following 3 cases are simulated.

Case 1: The sphere's density, ρ_s is much larger than the liquid's density, ρ_l ; i.e., $\rho_s \gg \rho_l$. In this case, it is assumed that $\rho_s = 8000 \text{ kg/m}^3$ and $\rho_l = 1300 \text{ kg/m}^3$.

Case 2: The sphere's density, ρ_s is close to the liquid's density, ρ_l but it is larger; i.e., $\rho_s \approx \rho_l$ but $\rho_s > \rho_l$. In this case, it is assumed that $\rho_s = 1350 \text{ kg/m}^3$ and $\rho_l = 1300 \text{ kg/m}^3$.

Case 3: The sphere's density, ρ_s equals the liquid's density, ρ_l ; i.e., $\rho_s = \rho_l$. In this case, it is assumed that $\rho_s = \rho_l = 1300 \text{ kg/m}^3$.

The natural frequency and the damping ratio can be

computed using (10) and (11) respectively, and are shown in Table I, II and III. The return time, T_r is defined as the time required for the sphere to enter and stay within the band of 3% set around the equilibrium position after the sphere is released. The return time, T_r can then be measured from the plots of the displacement versus time.

 TABLE I: REDUCTIONS OF NATURAL FREQUENCY, ω_n AND DAMPING

 RATIO, ζ DUE TO ADDED MASS EFFECTS FOR CASE 1

Parameters	Added Maas Not Included	Added Mass Included	% Reduction
Natural Frequency (rad/s)	437.0	420.3	3.8
Damping Ratio	0.103	0.099	3.8

 TABLE II: REDUCTIONS OF NATURAL FREQUENCY, ω_n AND DAMPING

 RATIO, ζ DUE TO ADDED MASS EFFECTS FOR CASE 2

Parameters	Added Maas Not Included	Added Mass Included	% Reduction
Natural Frequency (rad/s)	1063.8	874.0	17.8
Damping Ratio	0.251	0.206	17.8

 TABLE III: REDUCTIONS OF NATURAL FREQUENCY, ω_n AND DAMPING

 RATIO, ζ DUE TO ADDED MASS EFFECTS FOR CASE 3

Parameters	Added Maas Not Included	Added Mass Included	% Reduction
Natural Frequency (rad/s)	1084.1	885.2	18.4
Damping Ratio	0.256	0.209	18.4

The plots generated by Matlab in the coordinate axes are compared for the case with/without added mass for the 3 cases. The plots of the displacement versus time expressed in (12) are shown in Fig. 3, Fig. 7, and Fig. 11. The band of 3% set around the equilibrium position is represented by the 2 horizontal dash lines. The centers of the small circles on the graph mark and measure the return time. The return time is shown in Table IV.

The plots of the forces versus time expressed in (15),(16), and (17) for each case are shown in Fig. 4-Fig. 6, Fig. 8-Fig. 10, Fig. 12-Fig.14.

Fig. 3. Displacement(m) versus time(s), plot for Case 1.

Fig. 4. Spring force (N) versus Time(s), plot for Case 1.

Fig. 5. Damping force (N) versus Time(s), plot for Case 1.

Fig. 6. Inertia force (N) versus time(s), plot for Case 1.

Fig. 7. Displacement(m) versus time(s), plot for Case 2.

Fig. 8. Spring force (N) versus time(s), plot for Case 2.

Fig. 9. Damping Force (N) versus time(s), plot for Case 2.

Fig. 10. Inertia Force (N) versus time(s), plot for Case 2.

Fig. 11. Displacement(m) versus time(s), plot for Case 3.

Fig. 12. Spring Force (N) versus time(s), plot for Case 3.

Fig. 13. Damping Force (N) versus time(s), plot for Case 3.

Fig. 14. Inertia Force (N) versus time(s), plot for Case 3.

TABLE IV: RETURN TIME (s), T_r TO ORIGINAL EQUILIBRIUM POSITION

Case	No Added Mass	Added Mass	% Increase
1	0.0801	0.0835	4.2
2	0.0129	0.0192	48.8
3	0.0123	0.0184	49.6

V. DISCUSSION

The added mass is included in the modeling to obtain a more realistic model. The model obtained and the simulation show that as shown in Table I, II, and III, the natural frequency is reduced and the damping ratio is also reduced by the added mass; as shown in Fig. 3, Fig. 7, and Fig. 11, with the added mass included in the model, the sphere oscillates with the lower frequency and the higher amplitude when

compared to the case without the added mass. With the reduction in the damping ratio by the added mass, the sphere can oscillate with the higher amplitude.

The degree of the added mass effect depends on the difference between the sphere's density, ρ_s and the liquid's density, ρ_l ; i.e., the effect due to the added mass is small if the difference on the densities is large and the effect due to the added mass is large if the difference on the densities is small as it can be seen in Tables I, II, and III. For example, in case 1 where the sphere's density, ρ_s is much larger than the liquid's density, ρ_l , the % reduction in the natural frequency and the damping ratio is only 3.8%; in case 2 where the sphere's density, ρ_s is close to the liquid's density, ρ_l but it is larger, the % reduction in the natural frequency and the damping ratio is 17.8%, which is much higher compared to case 1. Therefore, the added-mass effect can be very significant if the body's density is higher than and close to the liquid's density and it can not be neglected in the modeling.

The return time for each case is shown in Table IV. With the added mass included in the model, the sphere takes longer time to return to the original equilibrium position and settles down, compared to the case without the added mass because the body with more inertia has more resistance to acceleration or deceleration and it will take longer time to come to rest at the equilibrium, compared to the body with a smaller mass.

The added-mass effect on the increase in the return time depends on the difference between the sphere's density, ρ_s and the liquid's density, ρ_l ; i.e., the added-mass effect on the return time is small if the difference on the densities is large and the added-mass effect on the return time large if the difference on the densities is small as it can be seen in Table IV. For example, in case 1 where the sphere's density, ρ_s is much larger than the liquid's density, ρ_l , the % increase in the return time only 4.2%; in case 2 where the sphere's density, ρ_s is close to the liquid's density, ρ_l but it is larger, the % increase in the return time is 48.8%, which is much higher compared to case 1.

Similarly, the added mass expectedly affects the spring force, the damping force, and the inertia force as the plots are shown in Fig. 4-Fig. 6, Fig. 8-Fig.10, Fig. 12-Fig. 14.

VI. CONCLUSION

This article presents the modeling and simulations of a sphere that oscillates vertically in a high-viscosity liquid. In general, the added mass is not considered in modeling an oscillator. In this article, a sphere is set into oscillation in a high-viscosity liquid where the oscillation can be assumed to take place in the Stokesian regime where the oscillation is linearly damped. The added mass due to the surrounding liquid is significant and needs to be taken into account in the modeling to obtain a more realistic model.

The main contribution of this article is to model and simulate the system and to show that i) the natural frequency of oscillation is reduced by the added mass; ii) the damping ratio is also reduced by the added mass; iii) the return time to

the original equilibrium is increased by the added mass; iv) furthermore, the difference on the densities between the liquid's density and the sphere's density dictates the degree of the added-mass effect; i.e., the effect due to the added mass is small if the difference on the densities is large and the effect due to the added mass is large if the difference on the densities is small. Therefore, the added-mass effect can be very significant if the body's density is higher than and close to the liquid's density and it can not be neglected in the modeling.

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