Estimation of Spectrum Eigenvalues Dynamic System on the Basis of Application Lyapunov Exponents

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***Abstract*—The method is based on identifications of a particular solution of the linear dynamic system on a class of static models with the dynamic specification on an input. On the basis of a particular solution the common decision are search. The common decision is a basis for an estimation of a spectrum system. For identification of a spectrum eigenvalues we introduced special structures, who described a modification of Lyapunov exponents. We gave generalization of the offer methods on the linear non-stationary dynamic systems. The algorithm of decision-making about a type of eigenvalues system is developed. We analyse properties of the offered special functions.**

***Index Terms*—Identification, lyapunov exponent, spectrum of eigenvalues, structure.**

I. Introduction

Spectrum of eigenvalues often applied to an estimation of quality dynamic system. Parametrical methods of identification in a combination to numerical methods are applied to a spectrum estimation. Very often for an estimation of a stability complicated systems Lyapunov exponents (LE) use. In [1] the algorithm of an evaluation largest LE (LLE) on the basis of the analysis of time series modifications of Dow Jones indexes stock market is offered. LLE is used for a proof of a chaotic modification of Dow Jones indexes. In [2] the analysis of methods estimation LE of the distributed systems on the basis of application known methods of an orthogonalization is made. Methods are based on preliminary deriving of trajectories movement system by a solution of the specify equation of a modification dynamics. In [3] it is offered to use LE for testing of condition dynamic systems with one equilibrium state. Authors for definition LE use interpolational methods of the analysis and handling of time series. Application of a graphic mode definition the LLE is given in work [4]. In [5] the problem an estimation of origin chaos in dynamic system on the basis of a calculation the largest LE is considered. For deriving estimation LLE is analyzed a time series. Other approaches to estimation LE on bases of deriving a solution dynamic system or fulfillment of the analysis experimental data are described in [6]-[8].

The analysis of scientific publications showed that indexes of stability were applied to an estimation of quality work dynamic system. For their evaluation or an estimation the full a priori information on system is used. In particular, various procedures of handling the time series describing a modification of processes are applied to a calculation of Lyapunov exponents. Researches according to quality of work dynamic system in the conditions of uncertainty or the incomplete a priori information on system on the specified set of indicators were not fulfilled. But there is a whole class of systems when origin of such situation is possible. In the conditions of lack full the information on a mathematical formulation application of the specified methods demands fulfillment of preliminary identification of system. Here we have some problems. The problem becomes complicated, if the information in a condition of normal operation system is accessible to observation. The methods considered above demand the further development.

We offer the approached method of an estimation type an equilibrium state of system in the conditions of uncertainty. It is based on identification of the particular solution of system for the purpose general solution selection at some initial condition. For identification of the particular solution of system the special class of static models with the dynamic specification on an input is used. For the first time such approach has been offered in [9]. In work we give generalization of this approach for a case of linear dynamic systems. Then we make the analysis of the received time series and we receive estimations of a spectrum eigenvalues (SE) linear dynamic system. For this purpose we form the time series describing a modification LE. For identification SE we introduce special structures [9], [10] for describe of modification LE. We apply a method of secants and we receive the approached estimations of a spectrum. The structure for identification LLE is offered.

II. Problem Statement

Consider linear dynamic system

$$\begin{aligned} \dot{X} &= AX + BU, \\ Y &= CX + DU, \end{aligned} \tag{1}$$

where $X \in R^m$ is a state vector, $U \in R^k$, $Y \in R^n$ are an input and an exit of system, $A \in R^{m\times m}$, $B \in R^{m\times k}$, $C \in R^{n\times m}$, $D \in R^{n\times k}$.

For (1) we have the experimental information

$$I_o = \{Y(t), U(t), t \in J = [t_0, t_1]\}. \tag{2}$$

Solution of system (1) we write as

$$X(t) = \mathcal{X}(t_0, U, t) \tag{3}$$

where *x* is an operator definitely defined by matrixes A, B.

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On the basis (2) from (3) we receive a solution of system (1) at $X_0 = X(t_0)$

$$X(t) = X_g(t) + X_q(t) \tag{4}$$

where $X_q(t)$ is a particular solution (1) with $U \in I_o$, $X_g(t)$ is a general solution (1) with $U(t) = 0$ at the unknown $X_0 \in I_o$. Designate $X_g(X_0, t)$ as the general solution (1) with $X_0 = X_0(Y_0) \in I_o$.

Problem: we should define estimations of a solution $X_g(t) = X_g(X_q, X_0, t)$ on set I_o and make a solution on a spectrum of eigenvalues system (1).

The problem consists in determination of estimation a solution $X_g(t) = X_g(X_q, X_0, t)$ on set I_o and decision-making on an equilibrium state.

The section headings are in boldface capital and lowercase letters. Second level headings are typed as part of the succeeding paragraph (like the subsection heading of this paragraph).

III. General Solution Definition

Procedure of a solution a problem is based on realization of two stages: identifications of a solution $X_q(t)$ and definition $X_g(t)$ as functions from $X_q(t)$. On a class of dynamic systems (1) selection of the separate particular solution is an arduous problem. Therefore for finding $X_q(t)$ we fulfil identification on a class of static models that is we search $X_q(t) = X_q(U, t)$ on some interval $J_q \subset J$. We can apply various approaches to a solution of this problem. The model choice depends on a frequency spectrum $Y(t)$.

We will explain a method of solution a problem for system (1) second order [9]. Consider system (1) with one input and an output. Designate $y = Y,\ u = U$ and $y \in R$, $u \in R$. Let $D_y(\omega)$, $D_u(\omega)$ are frequency spectrums u, y. We suppose $|y(t)| < \infty$, $|u(t)| < \infty$. Let $D_y(\omega) = D_u(\omega)$, that is the system (1) is linear and stationary. Root λ_i of a characteristic equation of system (1) will be negative: $\lambda_i \leq 0 \ \ \forall i = 1, 2$.

Write I_o as $I_o = I_o^q(J_q) \cup I_o^g(J_g)$, where $J_q \cup J_g = J \subseteq R$, I_o^q, I_o^q are the sets containing the information about X_q and X_g.

On set $I_o^q(J_q)$ we estimate the separate particular solution of system (1). As $x_1 = y \in R$, to deriving of a component $x_2 = \dot{x}_1$ of a vector X apply operation of differentiation a variable y. We will designate $\hat{x}_2 = \dot{y}$. To an estimation of the particular solution of system (1) apply static model

$$\hat{X}_q(t) = \hat{V} W(t) \tag{5}$$

where $\hat{V} \in R^{2\times 2}$ is a matrix of parameters model, $W = [u\ u']^T$.

For definition $\hat{V}$ we will use a least-squares method. Knowing $\hat{X}_q$, define $\hat{X}_g$

$$\hat{X}_g = \hat{X}_q - \hat{X} \tag{6}$$

where $\hat{X} = [y\ \hat{x}_2]^T$, $\hat{X}_g = [\hat{y}_g\ \hat{y}_g']^T = [\hat{x}_{g,1} - \hat{x}_{g,2}]^T$.

IV. Lyapunov Exponents

We will apply Lyapunov exponents to estimation SE of dynamic system (1). LE are a stability criterion of system and for a real-valued function $h(t)$ are defined as

$$\chi[h] = \overline{\lim_{t\to\infty}} \frac{\ln|h(t)|}{t} \tag{7}$$

where $\overline{\lim_{t\to\infty}}$ is a superior limit.

χ_i $\left(i = \overline{1, m}\right)$ a nontrivial solution of system (1) coincides with real parts of eigenvalues λ_i a matrix A. We will apply LE to definition of type an equilibrium point of system (1). Let the estimation of the general solution $X_g(t)$ $\forall t \in J_g$ of system (1) is known. We assume that the system is stable. Apply (7) to $\hat{y}_g(t)$ and receive

$$\chi[\hat{y}_g] = \overline{\lim_{t\to\bar{t}}} \frac{\ln|\hat{y}_g(t)|}{t} \tag{8}$$

where $\bar{t} \in J_g$ is an upper boundary on an interval $J_g \subset J$.

If the limit (7) exists, $\chi[\hat{y}_g]$ we have an estimation of a maximum eigenvalue of a matrix A. Hence, $\chi[\hat{y}_g]$ is an estimation of degree a stability system Eq. (1). If $m = 2$, for $\dot{\hat{y}}_g$ receive

$$\chi[\dot{\hat{y}}_g] = \overline{\lim_{t\to\bar{t}}} \frac{\ln|\dot{\hat{y}}_g|}{t}$$

On a basis $\chi[\hat{y}_g]$, $\chi[\dot{\hat{y}}_g]$ we make a solution on type of an equilibrium point system. For an increase of accuracy a calculation of LE we can use identification methods.

The idea of application Lyapunov exponents in identification problems is stated in [9], [10]. The offered approach is based on the analysis of coefficient structural properties (CSP) [10]. We give development of the given method. We will show at first relationship between CSP and LE.

V. Coefficient Structural Properties

Calculate for system (1) indicator

$$\rho(\hat{y}_g) = \ln|\hat{y}_g| \quad \forall t \in \bar{J}_g \subset J_g,$$

where $\bar{J}_g = [t_0, \bar{t}]$ define on the basis (8).

For an estimation of structural properties of this system introduce CSP [9]

$$k_s(t, \rho) = \frac{\rho(\hat{y}_g)}{t}. \tag{9}$$

We define interdependence between LE and CSP on set $\mathrm{I}_\rho = \{\rho(\hat{y}_g(t)),\ t \in \bar{J}_g\}$. We suppose that the system (1) is stable, that is $\mathrm{Re}(\lambda_i) \le 0$, $i = \overline{1,m}$, where $\lambda_i \in \sigma(A)$.

Problem: on the basis of the analysis of set I_ρ estimate a spectrum of eigenvalues $\sigma(A)$ a matrix A of system (1).

Problem solution reduces to realization of following stages.

1) Definition of dimension a spectrum $\sigma(A)$.
2) Deriving of initial estimations for a matrix spectrum A.
3) Working out of adaptive algorithm for an improvement of the estimations received at a stage 2. The adaptive algorithm and determination of dimensionality a spectrum $\sigma(A)$ was describe in [9].

Therefore stages 1 and 3 we done not consider. At a solution of a problem 2 we consider type of roots dynamic system. We will consider some cases.

VI. Determination of Spectrum Eigenvalues System

Let matrix eigenvalues A are arranged in decreasing order $\lambda_1 > \lambda_2 > \ldots > \lambda_m$. On the basis of (8), (9) define an estimation of a largest eigenvalue λ_1 of a matrix A

$$\chi[\hat{y}_g] = \lambda_1 = \frac{\overline{\lim\limits_{t \to \bar{t}}} |\hat{y}_g(t)|}{t}. \tag{10}$$

The main problem of application (10) is a choice of the moment $\bar{t}$. For it the solution are some approaches. The one approach is described in [9]. It is based on a method of secants for $\rho(\cdot)$. We will describe a method, which is based on the analysis CSP Eq. (9).

Consider informational set

$$\mathrm{I}_{\mathcal{LE}} = \left\{k_s\left(t, \rho\left(\hat{y}_g\right)\right), k_s\left(t, \rho\left(\dot{\hat{y}}_g\right)\right)\right\}$$

where $k_s\left(t, \rho\left(\hat{y}_g\right)\right)$ is CSP (9). The set $\mathrm{I}_{\mathcal{LE}}$ contained the information on a modification $\chi[\hat{y}_g]$. Consider map $\Gamma_{k_\rho} : k_s\left(t, \rho\left(\hat{y}_g\right)\right) \to k_s\left(t, \rho\left(\dot{\hat{y}}_g\right)\right)$ and structure $S_{\mathcal{LE}}$, correspond to it.

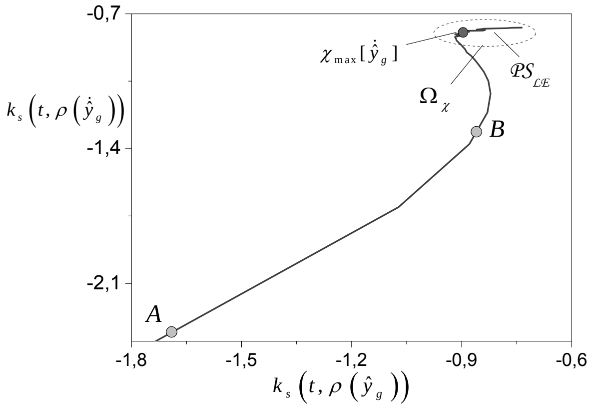

Fig. 1. $S_{\mathcal{LE}}$-structure.

For system of the second order with simple roots a structure $S_{\mathcal{LE}}$ instance display on Fig. 1.

Properties of structure $S_{\mathcal{LE}}$.

1) The fragment $\mathcal{PS}_{\mathcal{LE}}$ of structure $S_{\mathcal{LE}}$ contained the information on LLE $\chi_1[\dot{\hat{y}}_g]$.
2) The segment AB contained the data for decision-making about $\chi_i[\dot{\hat{y}}_g]$, $i \ge 2$.

$S_{\mathcal{LE}}$ for system a second order shows in the Fig. 1. After definition $\chi_1[\hat{y}_g]$ apply the approach offer in [9]. $\rho(t)$ present as

$$\rho(t) \cong \ln(2c_1) + \lambda_1 t + \ln(d_{12}) + 0,5\Delta\lambda_{12} t, \tag{11}$$

where $d_{12} > 0$, $\Delta\lambda_{12} = \lambda_2 - \lambda_1$. Divide (11) on t and on an interval $J_2 \subset J$ receive $\delta_2 = S_2(\rho(t))\big|_{t \in J_2} - \lambda_1 = 0,5\Delta\lambda_{12}$, where $S_2(\rho(t))$ is the average value $\rho(t)$, S_2 is an operator of smoothing on an interval J_2. Then

$$\lambda_2 = 2\delta_2 + \lambda_1.\quad \delta_2 < \infty \tag{12}$$

Apply (12) for an evaluation of other elements a spectrum of a matrix A. A choice of an interval J_i and values δ_i fulfill on the basis of the analysis CSP [9].

For complex and multiple roots we apply a method from [9] and structure $S_{\mathcal{LE}}$. We make a solution about type of roots on the basis of the analysis a variation a function

$$d(t) = \frac{\rho(\hat{y}_g)}{\rho(\dot{\hat{y}}_g)}.$$

Statement. Let a set I_g received as a result of handling experimental data I_o, and function $d(t)$. Then: i) if $d(t)$ a monotone function the system (1) has simple eigenvalues; ii) if function $d(t)$ contains a periodic component system (1) has complex eigenvalues; iii) if in some points $t \in J_g$ the condition of a monotonicity function $d(t)$ is not fulfilled, the system (1) has multiple eigenvalues.

VII. Conclusions

We solved a problem of an estimation a SE system on the basis of the analysis a time series. Unlike existing approaches we introduce special space and the map describing modification of Lyapunov exponents. The method of the approached identification set Lyapunov exponents on the basis of the analysis this map is offered. The further development of the described approach is application of the offered structures $S_{\mathcal{LE}}$ for an estimation of all spectrum dynamic system. On the basis of the analysis $S_{\mathcal{LE}}$ we will receive type of roots system and a spectrum of eigenvalues. We will give generalization of the given approach on nonstationary systems.

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