Variational Iteration Method for Transverse Vibrations of the Elastic, Tensioned Beam

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Abstract—In this study, the dynamic behavior of an axially accelerating, elastic, tensioned beam are determined using variational iteration method. The axial velocity is considered as a constant mean velocity. Analytical solutions for these types of problems are exactly limited in some cases. Therefore, we need numerical methods to analyse such problems. For the solutions of linear or nonlinear problems, variational iteration method is introduced as an effective analytical approximation technique. It is concluded that proposed technique is an efficient tool for the analysis of specified problems for which an analytical solution does not exist.

Key words—Axially moving tensioned beam, dynamic behavior, variational iteration method.

I. INTRODUCTION

Axially moving beam-typed structures attract attention of many scientists because of technical importance and applications widely in the engineering. The many intensive studies focus on the dynamic analysis of an elastic or viscoelastic beam axially moving with variable or constant velocity. Firstly, Mote [1] investigated the natural frequencies of an axially moving elastic beam. Mote [2] considered axially moving materials in the early literature. Besides, Ulsoy [3] and Wickert [4] *et al.* presented the studies on this matter. More recently, Öz and Pakdemirli [5] investigated the dynamic response of an elastic, tensioned beam with a constant mean velocity varying harmonically. Wickert and Mote [6] analyzed free non-linear vibration of an axially moving, elastic, tensioned beam over the sub- and supercritical transport speed ranges. Öz *et al.* [7] presented the transition behavior from string to beam such that the flexural stiffness vanished in the governing equation in [5]. Chen and Zhao [8] considered free nonlinear transverse vibrations for axially moving beams with pinned or clamped ends. Pellicano and Zirilli [9] examined the dynamic behavior of an axially moving beam with vanishing flexural stiffness and weak non-linearities.

The dynamic behaviors of axially moving beams are investigated by various analytical and numerical approaches.

Principal parametric resonances, combination resonances and stability boundaries for an axially accelerating, elastic, tensioned beam were discussed by Öz and Pakdemirli [5] via perturbation analysis. Öz [10] also used method of multiple time scales for the solution of an axially moving.

Euler–Bernoulli beam under simply supported and clamped boundary condition. Chen *et al.* [11] analysed transverse parametric vibration of an axially accelerating viscoelastic tensioned beam of which the material is defined by the Kelvin model by using Galerkin discretization. Chen *et al.* [12] introduced a modified finite difference approach to analyse transverse vibrations of an axially moving string. Čepón and Boltežar [13] applied Galerkin finite-element method to solve the governing equation of axially moving pre-tensioned beam with arbitrary support excitations. Wang *et al.* [14] used a collocation method to solve axially moving beam problems.

More recently, the some numerical approaches is alternative introduced to solve these type of problems. Lai *et al.* [15] applied Adomian decomposition method (ADM) for free vibration problems of Euler–Bernoulli beam under various supporting conditions. The natural frequencies and mode shapes of an Euler–Bernoulli beam under various supporting conditions were obtained by Liu *et al.* [16] via variational iteration method (VIM). Younesian *et al.* [17] examined the dynamic behavior of an elastic beam based on a nonlinear foundation by applying VIM.

In this study, the variational iteration method is applied to solve the Euler-Bernoulli beam problem [5]. The beam which is simply supported at both ends has a constant mean velocity. The more terms which can be obtained by VIM better represent the real behaviour of the system than other method.

II. THE EQUATION OF MOTION

The dimensionless equation of motion as given by Wickert [5], [6] is

$$\ddot{u} + 2\dot{u}'v + u'\dot{v} + v_f^2 u^{iv} + (v^2 - 1)u'' = 0 \quad (1)$$

where u is the transverse displacement, v is the axial velocity and v_f^2 is the dimensionless flexural stiffness which is a constant. The material particle of travelling beam experience local $\ddot{u}$, Coriolis $2\dot{u}'v$ and centripetal $v^2 u''$ acceleration components. Since the velocity is assumed as constant in governing equation, the term $u'\dot{v}$ vanishes [6]. Therefore, we get

$$\ddot{u} + 2\dot{u}'v + v_f^2 u^{iv} + (v^2 - 1)u'' = 0 \quad (2)$$

$$u(0, t) = u(1, t) = 0, \quad u''(0, t) = u''(1, t) = 0 \quad (3)$$

The dot denotes differentiation with respect to time t and the primes denote differentiation with respect to the spatial variable x .

III. VARIATIONAL ITERATION METHOD

In this part, the idea of variational iteration method is shortly introduced. Consider the general nonlinear differential equation given in the form

$$Lu(t) + Nu(t) = g(t) \quad (4)$$

where L is a linear operator, N is nonlinear operator, and $g(t)$ is a known function. By using the variational iteration method, a correction functional can be formed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda (Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)) d\xi \quad (5)$$

where λ is a general Lagrange multiplier, which can be determined optimally via variational theory; the subscript n means the n th approximation; $\tilde{u}_n$ is a restricted variation and $\delta\tilde{u}_n = 0$ [16].

IV. THE APPLICATION OF VIM

The correction functional can be written as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(s) \left\{ \ddot{u}_n(x, s) + 2v\dot{u}_n'(x, s) + v_f^2 \ddot{u}_n^{iv}(x, s) + (v^2 - 1)\ddot{u}_n''(x, s) \right\} ds \quad (6)$$

For the correction functional stable, if we apply the variational iteration method

$$\begin{aligned} \delta u_{n+1}(x, t) &= \delta u_n(x, t) + \delta \int_0^t \lambda(s) \left\{ \ddot{u}_n(x, s) \right\} ds \\ &= \delta u_n(x, t) + \int_0^t \left[\frac{\partial^2}{\partial s^2} \lambda(s) \right] \delta u_n(x, s) ds - \frac{\partial}{\partial s} \lambda(s) \delta u_n(x, s) \Big|_{s=t} + \lambda(s) \delta u_n'(x, s) \Big|_{s=t} \\ &= \left(1 - \frac{\partial}{\partial s} \lambda(s) \Big|_{s=t} \right) \delta u_n(x, t) + \lambda(s) \Big|_{s=t} \delta u_n'(x, t) + \int_0^t \left[\frac{\partial^2}{\partial s^2} \lambda(s) \right] \delta u_n(x, s) ds = 0 \end{aligned} \quad (7)$$

is obtained. Then, Euler-Lagrange equation is found as

$$\lambda''(s) = 0 \quad (8)$$

where initial condition is

$$\lambda(s) \Big|_{s=t} = 0, \quad 1 - \lambda'(s) \Big|_{s=t} = 0. \quad (9)$$

If the problem is solved, we obtain

$$\lambda(s) = s - t. \quad (10)$$

Substituting Lagrange multiplier into Eq. (6), the iteration formulation is obtained as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t (s - t) \left\{ \ddot{u}_n(x, s) + 2v\dot{u}_n'(x, s) + v_f^2 \ddot{u}_n^{iv}(x, s) + (v^2 - 1)\ddot{u}_n''(x, s) \right\} ds \quad (11)$$

The initial approximation can be assumed as

$$u_0(x, t) = X(x) \exp(i\omega t) \quad (12)$$

For $n=0$, substituting Eq. (12) into Eq. (11), we get

$$u_1(x, t) = X(x) \exp(i\omega t) + \int_0^t (s - t) \left\{ [-\omega^2 X(x) + 2iv\omega X'(x) + v_f^2 X^{iv}(x) + (v^2 - 1)X''(x)] \exp(i\omega s) \right\} ds \quad (13)$$

Result of the necessary process, we obtain the second term as

$$u_1(x, t) = X \exp(i\omega t) + \left[v_f^2 X^{iv} + (v^2 - 1)X'' + 2iv\omega X' - \omega^2 X \right] \frac{(\exp(i\omega t) - i\omega t - 1)}{\omega^2} \quad (14)$$

The iteration can be continued in this manner. Thus, the general solution of Eq. (2) is

$$u(x, t) = u_0(x, t) + u_1(x, t) + \dots \quad (15)$$

Since Eq. (12) holds Eq. (2), we obtain

$$v_f^2 X^{iv}(x) + (v^2 - 1)X''(x) + 2iv\omega X'(x) - \omega^2 X(x) = 0 \quad (16)$$

where initial conditions are

$$X(0) = X(1) = X''(0) = X''(1) = 0. \quad (17)$$

Then, the solution of Eq. (16) is

$$X(x) = c_1 e^{ir_1 x} + c_2 e^{ir_2 x} + c_3 e^{ir_3 x} + c_4 e^{ir_4 x} \quad (18)$$

where r_n ($n=1, \dots, 4$) satisfies dispersive relation

$$v_f^2 r_n^4 + (1 - v^2) r_n^2 - 2v\omega r_n - \omega^2 = 0, \quad n=1, \dots, 4 \quad (19)$$

Applying the simply supported boundary conditions, we obtain

$$c_1 + c_2 + c_3 + c_4 = 0 \quad (20)$$

$$c_1 r_1^2 + c_2 r_2^2 + c_3 r_3^2 + c_4 r_4^2 = 0 \quad (21)$$

$$c_1 e^{ir_1} + c_2 e^{ir_2} + c_3 e^{ir_3} + c_4 e^{ir_4} = 0 \quad (22)$$

$$c_1 r_1^2 e^{ir_1} + c_2 r_2^2 e^{ir_2} + c_3 r_3^2 e^{ir_3} + c_4 r_4^2 e^{ir_4} = 0 \quad (23)$$

The support condition is found by nontrivial solutions of Eqs. (20)-(23)

$$\begin{aligned} & \left(e^{i(r_1+r_2)} + e^{i(r_3+r_4)} \right) (r_1^2 - r_2^2) (r_3^2 - r_4^2) + \left(e^{i(r_1+r_3)} + e^{i(r_2+r_4)} \right) (r_2^2 - r_4^2) (r_3^2 - r_1^2) \\ & + \left(e^{i(r_2+r_3)} + e^{i(r_1+r_4)} \right) (r_1^2 - r_4^2) (r_2^2 - r_3^2) = 0 \end{aligned} \quad (24)$$

ω and r_n can be numerically calculated by using the dispersive relation and the support condition. The coefficients are obtained by elimination from Eqs. (20)-(23). Finally, Eq. (18) are

$$\begin{aligned} X(x) = & c_1 \left\{ e^{ir_1x} - \frac{(r_4^2 - r_1^2)(e^{ir_3} - e^{ir_1})}{(r_4^2 - r_2^2)(e^{ir_3} - e^{ir_2})} e^{ir_2x} - \frac{(r_4^2 - r_1^2)(e^{ir_2} - e^{ir_1})}{(r_4^2 - r_3^2)(e^{ir_2} - e^{ir_3})} e^{ir_3x} \right. \\ & \left. + \left[-1 + \frac{(r_4^2 - r_1^2)(e^{ir_3} - e^{ir_1})}{(r_4^2 - r_2^2)(e^{ir_3} - e^{ir_2})} + \frac{(r_4^2 - r_1^2)(e^{ir_2} - e^{ir_1})}{(r_4^2 - r_3^2)(e^{ir_2} - e^{ir_3})} \right] e^{ir_4x} \right\} \end{aligned} \quad (25)$$

where r_1, r_2, r_3, r_4 are constant [18]. Finally, the general solution is obtained by substituting Eqs. (25) into Eq. (15).

Fig. 1. Variation of natural frequency with dimensionless velocity for the different modes. Stars, circles and squares represent first, second and third mode natural frequencies, respectively ($\nu_f = 0.8$).

Fig. 2. Comparisons of first natural frequency values for different flexural stiffness for first mode.

V. NUMERICAL RESULTS

In this section, the graphs for the natural frequencies will be introduced. The natural frequencies are obtained from the solution of Eq. (19) and (24).

As seen, the dimensionless velocity at which the fundamental natural frequency close to zero is defined as the critical velocity and is approximately equal to the value of 2.7 as shown in Fig. 1. On the other hand, natural frequencies increase with increasing flexural stiffness (Fig. 2).

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